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Ultrafilters and Ultraproducts across Mathematics, and Related Topics

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Monday 6th

Some new results about left ideals in βS

Dona Strauss
University of Hull

Left ideals in βS, where S denotes a discrete semigroup, play a significant role in topological dynamics. I shall present some new results by Neil Hindman and myself about their properties, including a discussion of semigroups S for which the minimal left ideals of βS are finite. I shall also discuss the possibility of the smallest ideal of βS being homeomorphic to a cartesian product of smallest ideals of Stone-Čech compactifications, and shall mention an extension of previously known results about the fact that, if S is countably infinite and cancellative, every non-minimal left ideal of βS contains many right cancelable elements of βS.

The block sets conjecture

Imre Leader
University of Cambridge

The block sets conjecture is a natural Hales-Jewett-type conjecture. Its main interest is that, if true, it would imply that all transitive subsets of Euclidean space are Ramsey. It is known only in very few cases.
In this talk we will present background on this conjecture, and then move on to some recent more results.
This is a joint work with Maria Ivan.

Image partition regular matrices and concepts of largeness

Neil Hindman

Howard University

Image partition regular matrices are important in Ramsey Theory. For example, van der Waerden's Theorem is the assertion that for each $l \in \mathbb{N}$, the $(l+1) \times 2$ matrix with rows $(1\ 0), (1\ 1), \ldots, (1\ l)$ is image partition regular over $\mathbb{N}$. Given a discrete semigroup $(S, +)$, there are a large number of notions of largeness that are determined by the algebra of the Stone-Čech compactification βS of S. Included among these are the notions of *central* and *piecewise syndetic*. We show that for several notions of largeness in a semigroup, if $u, v \in \mathbb{N}$, A is a $u \times v$ matrix satisfying restrictions that vary with the notion of largeness, and if C is a large subset of $\mathbb{N}$, then $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u\}$ is large in $\mathbb{N}^v$.
This is joint work with Dona Strauss.

Monochromatic products and sums in $\mathbb{N}$ and $\mathbb{Q}$

Matt Bowen

McGill University

We show that any 2-coloring of $\mathbb{N}$ and any finite coloring of $\mathbb{Q}$ contains monochromatic sets of the form $\{x, y, xy, x + y\}$, as well as generalizations involving several variables and arithmetic progressions. This is partially based on joint work with Marcin Sabok.

Partition Regularity of Affine Plane Curves

Paulo Arruda
University of Vienna

Exploring an interplay between ultrafilters, nonstandard analysis and basic algebraic geometry, we extend Rado's Theorem on two variables by proving that the only irreducible Diophantine equation in two variables that admits infinitely many monochromatic solutions is $x = y$. Working with an algebraically closed field, we also use a known fact about colorings related to functions with no fixed points to provide a bound on a number of colors needed to show that an equation in two variables does not admit infinitely many monochromatic solutions.

Subtracted image partition regular matrices

Sourav Kanti Patra
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A finite or infinite matrix A with entries from $\mathbb{Q}$ is image partition regular provided that whenever $\mathbb{N}$ is finitely colored, there must be some $\vec{x}$ with entries from $\mathbb{N}$ such that all the entries of $A\vec{x}$ are in some color class. There are several characterizations of finite image partition regular matrices. Compare to finite case infinite image partition regular matrices are very hard to analyze. Millliken-Taylor matrices and centrally image partition regular matrices are known examples of infinite image partition regular matrices. Here we introduce a new notion of image partition regularity. It will produce several new examples of infinite image partition regular matrices.

Monochromatic Solutions To $a + b = cd$ Near Zero

Md Moid Shaikh

Department of Mathematics, Maharaja Manindra Chandra College

In [4, Corollary 1], Sárközy proved that there exist $a \in \mathcal{A}$, $b \in \mathcal{B}$, $c \in \mathcal{C}$, and $d \in \mathcal{D}$ such that $a + b = cd$ with the conditions p is any prime, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D} \subseteq \mathbb{Z}_p$, and $|\mathcal{A}| \cdot |\mathcal{B}| \cdot |\mathcal{C}| \cdot |\mathcal{D}| > p^3$. In [2] Gyarmati and Sárközy extended this result to any finite field $\mathbf{F}_q$, where q is a prime power.

These results directed Csikvári, Sárközy, Gyarmati to raise a question [1, Problem B], "Does there exist a monochromatic solution to $a+b = cd$ with $a \neq b$, whenever the set of natural numbers $\mathbb{N}$ is finitely colored?" They were unable to answer this question but proved certain partial results such as for every finite partition of $\mathbb{N}$, the equation $a + b = cd$ with $a \neq b$ can be solved so that a and b, respectively c and d are in the same partition cell. In [3], Hindman answered this question affirmatively by showing, in addition, that one can demand that a, b, c, d are all distinct and the color of $a + b$ is the same as that of a, b, c, and d. In fact, he proved a considerably stronger result using the algebraic structure of $\beta\mathbb{N}$, the Stone-Čech compactification of $\mathbb{N}$. We want to establish the existence of monochromatic solutions to $a + b = cd$ with $a \neq b$ near zero for a dense subsemigroup of $((0, \infty), +)$, that is, we will get the above mentioned monochromatic solutions as small as we want. This is a joint work with Sourav Kanti Patra.

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Fragile and indesctructible ultrafilters on ω

David Chodounský
Institute of Mathematics of the Czech Academy of Sciences

We study ultrafilters on countable sets which are indestructible with the Sacks forcing, or in general generate an ultrafilter in some extensions of the universe of sets V. We say that an ultrafilter is fragile if it never generates an ultrafilter if new subsets of ω are added to the universe. It was proved by Bartoszyński, Goldstern, Judah, and Shelah that there exists an ideal $\mathcal{I}$ on ω such that an ultrafilter is fragile whenever it is disjoint with $\mathcal{I}$. We improve this result by showing that the ideal $\mathcal{Z}$ of subsets of ω with zero upper density also has this property.
This is a joint work with Osvaldo Guzmán and Michael Hrušák.

Tuesday 7th

Infinite sumsets in sets with positive density, Part 1

Joel Moreira

University of Warwick

In the 1970's Erdös asked several questions about what kind of infinite structures can be found in every set of natural numbers with positive density. Whilst many of these questions remain open, one conjecture was recently resolved in joint work with Richter and Robertson: it was proved that every set of natural numbers with positive density contains the sumset of two infinite sets. In this talk, which is the first of a two parts series, we present a natural generalization for Erdös' conjecture, stating that, for any $k \in \mathbb{N}$, any set of positive upper Banach density contains a sumset $B_1 + \cdots + B_k$, where all $B_1, \ldots, B_k \subseteq \mathbb{N}$ are infinite. I will describe some of the initial ideas that go into the proof of this new result, including a reduction motivated by ultrafilters to a result in topological dynamics. This talk is based on joint work with Kra, Richter and Robertson.

Infinite sumsets in sets with positive density, Part 2

Florian Karl Richter

EPFL

In this talk we will go deeper into the proof of a new result that asserts that every subset of the integers with positive density contains a sumset $B_1 + \cdots + B_k$ where $B_1, \cdots, B_k$ are infinite. Our method relies on a newfound connection between k-fold sumsets in the integers and return times of orbits in k-fold joinings of measure preserving systems arising from the Host-Kra structure theory. I will make an effort to keep the talk accessible to a wider audience by motivating the dynamical notions involved through simple examples. The talk will conclude with a hodgepodge of open problems, some connected to ultrafilters and the algebra on the Stone-Čech compactification of the integers. This talk is based on joint work with Kra, Moreira, and Robertson.

Some new applications of ultrafilters to Diophantine approximation, combinatorial number theory and ergodic theory

Vitaly Bergelson
Ohio State University

The goal of this talk is to review some recent results obtained jointly with Rigo Zelada.
In the first part of our talk we will focus the discussion on the set of the form $R(v, \epsilon) = \{n \in \mathbb{N} : \|v(n)\| < \epsilon\}$ where v is a real polynomial, with $\deg(v) \geq 1$, satisfying $v(0) = 0$. $\epsilon > 0$, and $\| \cdot \|$ denotes the distance to the nearest integer. It turns out that if v is an odd polynomial (i.e. $v(-x) = -v(x)$) the sets $R(v, \epsilon)$ have, somewhat unexpectedly, strong properties of largeness which are best described via the iterated sets of differences. We will discuss an ultrafilter approach to obtaining these results.
We will then describe the applications of new Diophantine results to ergodic theory and combinatorics. These include a new characterization of weakly mixing systems as well as a new invariant of Furstenberg-Sárközy theorem which states that for any integer-valued polynomial $v(n)$ satisfying $v(0) = 0$, and for any set $E \subseteq \mathbb{N}$ of positive upper density, there exist two distinct elements $x, y \in E$ and a positive integer n such that $x - y = v(n)$.

Khintchine's recurrence theorem

Dibyendu De
Department of Mathematics, University of Kalyani

Celebrated Khintchine's recurrence theorem improves the classical Poincaré recurrence theorem in two ways: it provides a lower bound on the size of the intersection and shows that the set of return times is large. Furstenberg multiple recurrence is a vast generalization of Poincaré recurrence theorem. Aiming at a simultaneous extension of Khintchine's and Furstenberg's recurrence theorems, Bergelson, Host and Kra established Khintchine's recurrence theorem for 2nd and 3rd order for Ergodic system of $\mathbb{Z}$ action.
Further Furstenberg and Katznelson generalized Furstenberg's recurrence theorem to multiple commuting transformations. It is therefore natural to ask whether a result analogous to multiple recurrence theorem for multiple commutating transformations in the direction of Khintchine recurrence theorem exists. Qing Chu proved the following generalization of two transformations: let $(X, \mathcal{B}, \mu)$ be a probability space, and let T_1 and T_2 be two commuting invertible measure-preserving transformations and that $(X, \mathcal{B}, \mu, T_1, T_2)$ is ergodic and if A is a set with positive measure then for every $\epsilon > 0$

$$\{n \in \mathbb{Z} : \mu(A \cap T_1^{-n} A \cap T_2^{-2n} A) > \mu(A)^4 - \epsilon\}$$

is syndetic. Here it is to note that the exponent 4 cannot be replaced by 3.
For the action of non commutative group the scenario is more complicated. Bergelson, McCutcheon and Zhang proved the following multiple recurrence theorem for two commutaing transformation for the action of countable amenable groups, which is known as non commutative Roth theorem.
In a recent work P. Durcik, R. Greenfeld, A. Iseli, A. Jamneshan, and J. Madrid extended non commutative Roth theorem for arbitrary amenable group action, on the other hand Qui Chu's version of Khintchine recurrence theorem was extended for arbitrary countable amenable group action. Keeping these two theorems in mind it is natural to expect a version of Khintchine recurrence theorem for the action of arbitrary amenable group.
In the present work we establish that such generalization holds.

On limit spaces and actions of groups

Cornelia Druţu Badea
Mathematical Institute, University of Oxford

Using ultrafilters one can construct a limit of a sequence of spaces, or of a sequence of actions of a group. Such limits are used in topology (compactification of the Teichmüller space), geometry (Gromov's Polynomial Growth Theorem, isoperimetric inequalities, rigidity results) and analysis (Banach limits and amenability, local theory of Banach spaces, Assouad's theorem on embeddings). The range of distinct limits for a given infinite group relates to the Continuum Hypothesis.

Integration with filters

Monroe Eskew
University of Vienna

In recent work, we introduced a notion of integration that allows for infinitesimal measures without needing the axiom of choice. The range of values of integrals of real functions is in general not a field but a "comparison ring", which is preserved under reduced powers. This notion of integral is able to represent classical integrals in a canonical way. As an application, we define a geometric measure over an infinite-dimensional vector space that overcomes some of the well-known limitations for real-valued measures and addresses an old paradox about conditional probability. This leads to a notion of fractal dimension, and we apply Martin's Axiom to explore the possible order structure among these dimensions.

A walk through some applications of ultrafilters and ultraspaces in Metric Fixed Point Theory

Maria Japón

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Metric fixed point theory has its origins with Banach's contraction principle in 1922: If (X, d) is a complete metric space and $T : X \to X$ is a contractive mapping (there is some $c \in (0, 1)$ such that $d(Tx, Ty) \leq cd(x, y)$ for all $x, y \in X$), then there is a unique fixed point.
Unlike Schauder's fixed point theorem for continuous functions, which has a topological flavour, the metric or distance plays an essential role when contractive mappings are considered. If we let T be nonexpansive ($d(Tx, Ty) \leq d(x, y)$ for all $x, y \in X$) the existence of a fixed point is no longer assured.
The first positive results for the existence of fixed points for nonexpansive mappings appeared in the 1960s when Browder and Ghöde independently proved that every nonexpansive self-mapping defined from a closed convex bounded of a Hilbert space H does have a fixed point. If we replace H by the sequence space ℓ_1 or c_0, the above statement is no longer true. At a first sight, it could be thought that the weak compactness of the domain may have something to say, but Alspach's example, published in 1981, strongly collided with this idea. In fact, Alspach proved that the baker transform is nonexpansive for the $\|\cdot\|_1$-norm and leaves invariant a convex weakly compact subset containing no fixed points.
There is a wide and extensive literature developed along the last 60 years studying connections between geometry in Banach spaces and the existence of fixed points for nonexpansive mappings (see some references below). Nevertheless, some long-standing classical problems dating back to the second half of the twentieth century are still open.
The purpose of this talk is to show how ultra-method techniques have become an essential tool applied quite successfully in many proofs concerning metric fixed point results, playing a twofold role: it allows us to simplify many long and complex arguments and simultaneously it has propelled the achievement of great advances within the theory.
Some open problems, where new insights concerning the use of ultrafilters and ultra-product spaces could be of application, will be displayed.

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Quasi-Følner sequences on arbitrary countable semigroups

Hedie Hosseini

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Let S be an arbitrary countable semigroup and let P be a countable subsemigroup of $(S^S, \circ)$, where S^S is semigroup of the collection of all functions $f : S \to S$ with composition operation. A sequence $\mathcal{F} = \{F_n\}_{n\in\mathbb{N}}$ of non-empty finite subsets of S is called quasi-Følner sequence on S if $\lim_{n\to\infty} \frac{|F_n \cap g(F_n)|}{|F_n|} = 1$, for every $g \in P$. We show that a suitable countable amenable subsemigroup P of S^S induces a quasi-Følner sequence on S, so the upper density $\overline{d}_{\mathcal{F}}$ and the upper Banach density $d^*_{\mathcal{F}}$ has been defined on S. Also, we prove a weak version of Furstenberg's correspondence principal Theorem on arbitrary countable semigroup S and prove some combinatorial results on arbitrary countable semigroups.

Wednesday 8th

Ultrafilters and topological Ramsey spaces

Natasha Dobrinen
Department of Mathematics, University of Denver

Topological Ramsey spaces are central to various structural results involving ultrafilters satisfying weak partition properties. Many partial orders used to build such ultrafilters have been found to contain topological Ramsey spaces densely inside of them, making available strong techniques which enable fine-tuned results. The first such spaces were built by Dobrinen and Todorcevic in order to classify the exact Rudin-Keisler and Tukey structures for a hierarchy of weak Ramsey ultrafilters constructed by Laflamme. This talk will survey ultrafilters with weak Ramsey properties and applications afforded by Ramsey space techniques to the Katětov, Rudin-Keisler, and Tukey orders, calculations of partition relations of ultrafilters, and properties of models of ZF with ultrafilters. Central to these results are Ramsey-classification theorems extending theorems of Erdős-Rado and Pudlák-Rödl providing canonical equivalence relations on fronts and barriers. Survey references for the talk: [1], [2].

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Nonstandard Analysis Without Ultrafilters

Guillaume Massas
University of California, Berkeley

Although nonstandard analysis has grown into a diverse field that studies a wide variety of structures, many of its applications to ordinary mathematics rely on the existence of a first-order structure $\mathscr{M}$ satisfying versions of the following axioms [1]:

1. (Extension Principle) The signature of $\mathscr{M}$ contains a nonstandard extension S^* for any finitary relation S on $\mathbb{R}$.

2. (Transfer Principle) $\mathbb{R}$ is an elementary substructure of $\mathscr{M}$.

3. (Saturation Principle) Any countable sequence of non-empty nested definable subsets of $\mathbb{R}^*$ has non-empty intersection.

It is well-known that the existence of a nonstandard extension satisfying these three conditions implies the existence of a non-principal ultrafilter on ω, and thus exceeds the resources of semi-constructive mathematics ($ZF + DC$), a natural setting for standard analysis [5, Chap. 14].
In this talk based on [4], we will present a way out of this problem that relies on an alternative to Tarskian semantics known as possibility semantics [2]. In particular, we show that the appeal to classical ultrapowers modulo a non-principal ultrafilter on ω can be avoided by using the partially-ordered set of all non-principal filters on ω to define a semi-constructive analogue ${}^\dagger\mathcal{R}$ of the hyperreal line which satisfies versions of the Extension, Transfer and Saturation Principles. We show how basic results of Robinsonian nonstandard analysis can be retrieved, and how ${}^\dagger\mathcal{R}$ can serve as an alternative foundation for infinitesimal calculus *à la* Keisler [3]. We conclude by highlighting connections with Boolean-valued analysis [6] and forcing in models of $ZF + DC$.

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Ultrapowers and topological groups

Michael Hrušák

Universidad Nacional Autónoma de México

We shall discuss the recent solution to Comfort's and van Douwen's problems on contably compact topological groups and related questions concerning ultrafilters.

Dense metrizable subspaces in powers of Corson compacta

Santi Spadaro
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A Corson compactum is a compact space which embeds into $\Sigma(\mathbb{R}^\kappa) = \{x \in \mathbb{R}^\kappa : |\{\alpha < \kappa : x(\alpha) \neq 0\}| \leq \aleph_0\}$, for some cardinal κ. Many natural classes of compacta lie in this class, including weakly compact subspaces of Banach spaces. The problem of when a Corson compactum has a dense metrizable subspace is a well-known one (it has a positive answer, for example, for the class of weakly compact subspaces of Banach spaces). In 1981 Todorcevic constructed the first ZFC example of a Corson compactum without a dense metrizable subspace.
The countable product of Corson compacta is still a Corson compactum and this inspired us to study the above problem in this setting. We characterize when the countable power of a Corson compactum has a dense metrizable subspace and construct consistent examples of Corson compacta whose countable power does not have a dense metrizable subspace. It is still open whether a Corson compactum with these features can be constructed in ZFC.
This is joint work with Arkady Leiderman and Stevo Todorcevic.

Pseudofiniteness and measurability in the everywhere infinite forest

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A structure M is said to be pseudofinite if every first-order sentence that is true in M has a finite model, or equivalently, if M is elementarily equivalent to an ultraproduct of finite structures. For this kind of ultraproducts, the fundamental theorem of ultraproducts (Los' Theorem) provides a powerful connection between finite and infinite structures, which can be used to prove qualitative properties of large finite structures using combinatorial methods applied to non-standard cardinalities of definable sets. Also, in the other direction, quantitative properties in classes of finite structures often induce desirable model-theoretic properties in their ultraproducts.

The concept of asymptotic classes of finite structures was defined by Macpherson and Steinhorn in [4] as classes of finite structures that satisfy strong conditions on the sizes of definable sets. The most notable examples are the class of finite fields, the class of cyclic groups, or the class of Paley graphs. The infinite ultraproducts of asymptotic classes are all supersimple of finite SU-rank, but recent generalizations of this concept (known as *multidimensional asymptotic classes*, or m.a.c.) are more flexible and allow the presence of ultraproducts whose SU-rank is possibly infinite. (cf. [1], [5]).
In this talk we will introduce the concepts of pseudofinite structures and multidimensional asymptotic classes having as motivating example the theory of the *everywhere infinite forest*, which is the theory of an acyclic graph G such that every vertex has infinite degree, which is a well-known example of an ω-stable theory of infinite rank.
We will show that G is also pseudofinite, and is elementarily equivalent to an infinite ultraproduct M of a multidimensional *exact* class of finite regular graphs. Moreover, we will present a precise description of the possible cardinalities of their definable sets in terms of polynomials in two variables with integer coefficients and evaluated in the non-standard cardinalities $\alpha = |M|$ and $\beta = |\{x \in M : xRa\}|$ (the non-standard cardinality of any ball of radius 1 in M). Combining these results it is also possible to give a description of forking and U-rank for the infinite everywhere forest in terms of certain *pseudofinite dimensions* (see [2],[3]).
This is joint work with Melissa Robles.

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Big Ramsey degrees of hypergraphs

Matěj Konečný

Department of Applied Mathematics (KAM), Charles University

We prove that unconstrained ultrahomogeneous relational structures (e.g. edge-coloured hypergraphs) have finite big Ramsey degrees if and only if they are ω-categorical, that is, have only finitely many n-types for every n (for hypergraphs this corresponds to having finitely many colours of each arity). The "if" part is proved by an application of the product version of Milliken's Tree Theorem for rapidly branching trees, the "only if" part follows by an adaptation of an argument that the Halpern–Läuchli theorem does not hold for infinitely branching trees.
This is the first time we were able to handle structures in infinite languages and our result shows that the breaking point seems to be when the so-called tree of types becomes infinitely branching. In the talk, we will introduce the area of big Ramsey degrees and outline the correspondence between big Ramsey results and tree Ramsey theorems which we use to prove our result.
This is joint work with Braunfeld, Chodounský, de Rancourt and Kawach.

Big Ramsey degrees and trees with sucessor operation

Jan Hubička

Department of Applied Mathematics (KAM), Charles University

Recent results on big Ramsey degrees (by Dobrinen on triangle-free and Henson graphs, by Zucker on free amalgamation classes) involve formulation of special purpose tree Ramsey theorems for trees with coding nodes. The proofs of such theorems are quite involved and follow the basic scheme of the Harrington's proof of the Milliken tree theorem via the method of forcing. The main technical difficulties come from the asymmetric placement of coding nodes and complicated definitions of subtrees which need to preserve structural properties.
A recent link to the Carlson–Simpson theorem offers a new direct approach to obtaining these results. We will discuss an abstract tree theorem for trees with a sucessor operation which can be used to show all known big Ramsey degrees on binary structures and generalises to some cases of structures of higher arity. It can be seen as a joint strengthening of the Milliken tree theorem for regular trees and the Carlson–Simpson theorem for trees with unbounded branching.
This is joint work with Balko, Chodounský, Dobrinen, Konečný, Nešetřil, Vena and Zucker.

An Algebraic Hypergraph Regularity Lemma

Alexis Chevalier

University of Oxford

In his 2012 paper 'Expanding polynomials…', Tao proves the algebraic regularity lemma, which strengthens the classical Szemerédi regularity lemma to definable graphs in finite fields. The algebraic regularity lemma improves the Szemerédi regularity lemma by providing regular decompositions of the definable graphs which have no irregular pairs and such that the error bounds on regularity vanish as the size of the finite field grows.
Tao asks if the algebraic regularity lemma can be extended to definable hypergraphs. We answer this question positively by giving a new analysis of the algebraic regularity lemma. We use the model theory of

pseudofinite fields to relate the combinatorial notion of regularity (for graphs and for hypergraphs) to Galois-theoretic information associated to definable sets. We will see that with this new analysis in hand, the algebraic hypergraph regularity lemma is a natural extension of Tao's algebraic regularity lemma.

Friday 10th

Iterated Ultrapowers and Nonstandard Proof of Szemerédi's Theorem

Renling Jin

College of Charleston

Iterated ultrapowers of the standard universe create a nonstandard universe with many levels of infinitely large integers and various elementary embeddings. With the help of these infinitely large integers and elementary embeddings one can simplify some lengthy standard combinatorial arguments as well as discover some new theorems or phenomena. We will present a few known and new examples to demonstrate how the iterated ultrapower technique is used in the proofs of combinatorial results including a nonstandard proof of Szemerédi's theorem which simplifies further the Tao's recent simplification of Szemerédi's original arguments.

Some thoughts on the use of nonstandard methods to extend Roth's Theorem

Steven Leth

University of Northern Colorado

Jin's recent proof of Roth's and Szemerédi's theorem illustrates how the use of nonstandard methods can allow us to replace complicated combinatorial arguments with more elementary and more accessible ones. In this talk I will explore some possibilities and some challenges in attempting to use nonstandard methods to extend Roth's theorem to sparser sets. Recently Bloom and Sisask have shown that any subset of the natural numbers with no 3-term arithmetic progression must have asymptotic density less than $\frac{1}{\log(n)^{1+c}}$ for some constant $c > 0$, providing the strongest generalization so far, and they conjecture that this upper bound can be significantly improved. I will outline some ways in which a nonstandard approach might allow us to obtain similar or stronger results.

Dynamics of monoids and infinite dimensional Ramsey Theorems

Sławomir Solecki
Cornell University

We present a general theorem on dynamics of actions of monoids by endomorphisms of semigroups. We explain how this theorem for differents monoids gives, as consequences, different infinite dimensional Ramsey Theorem for sequences.

Colorful actions of monoids, finite and infinite

Claudio Agostini
Università degli Studi di Torino, Dipartimento di Matematica "G. Peano"

Carlson's theorem on variable words [3] and Gowers' FIN_k theorem [5] are generalizations of Hindman's theorem [6] that involve a monoid action on a semigroup. In short, they state that for any finite coloring of a semigroup there is an infinite monochromatic "span set". They differ in the choice of the monoid.
In [7], Solecki introduced the notion of Ramsey monoid to isolate the common underlying structure of these theorems. Then, he proved a number of results, among which some necessary and sufficient conditions for a monoid to be Ramsey. In [2], we provided a purely algebraic characterization of finite Ramsey monoids and gave some new sufficient and necessary conditions for a monoid to be $\mathbb{Y}$-controllable — a more technical notion that refines the concept of Ramsey monoid and allows to generalize other theorems in combinatorics, like the Furstenberg-Katznelson Ramsey Theorem [4]. In [1], we introduce a new local version of Ramsey monoids and $\mathbb{Y}$-controllable monoids, and then we prove a number of results. Among them, we complete the characterization of Ramsey monoids by proving that all Ramsey monoids are finite,

and we provide instead examples of infinite $\mathbb{Y}$-controllable monoids.
In this talk, I will introduce these different notions and explain the known relations between them and with other algebraic classes of monoids.
This is a joint work with Eugenio Colla.

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A Ramsey Characterisation of Eventually Periodic Words

Maria-Romina Ivan
University of Cambridge

A factorisation $x = u_1u_2\cdots$ of an infinite word x on alphabet X is called 'super-monochromatic', for a given colouring of the finite words X^* on alphabet X, if each word $u_{k_1}u_{k_2}\cdots u_{k_n}$, where $k_1 < \cdots < k_n$, has the same colour. A direct application of Hindman's theorem shows that if x is eventually periodic, then for every finite colouring of X^*, there exists a suffix of x that admits a super-monochromatic factorisation. What about the converse?
In this talk we show that the converse does indeed hold: thus a word x is eventually periodic if and only if for every finite colouring of X^* there is a suffix of x having a super-monochromatic factorisation. This has been a conjecture in the community for some time. Our main tool is a Ramsey result about alternating sums. This provides a strong link between Ramsey theory and the combinatorics of infinite words.
This is a joint work with Imre Leader and Luca Q. Zamboni.

Large strongly anti-Urysohn spaces from weak P-point ultrafilters

István Juhász
Alfréd Rényi Institute of Mathematics, ELKH

A Hausdorff space is *strongly anti-Urysohn* (in short: SAU) if it has at least two non-isolated points and any two *infinite* closed subsets of it intersect.
In joint work with Shelah, Soukup and Szentmiklóssy [2], we answered the main question of [1] by providing a ZFC construction of a locally countable SAU space of cardinality $2^{\mathfrak{c}}$. The construction, that I will sketch in my talk, hinges on the existence of $2^{\mathfrak{c}}$ weak P-point ultrafilters in ω^*, a very deep result of Ken Kunen.
A further non-trivial modification of this example yields a SAU space in which all non-empty open sets have cardinality $2^{\mathfrak{c}}$.
It, however, remains an open question if SAU spaces of cardinality $> 2^{\mathfrak{c}}$ could exist, while it was shown in [1] that $2^{2^{\mathfrak{c}}}$ is an upper bound.

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Structural Ramsey theory and free superpositions; a model-theoretic perspective

Nadav Meir

University of Wroclaw

There are several well-known constructions of new Fraïssé classes from existing ones; the full product and the free superposition are examples of such constructions which preserve the Ramsey property. (The former is classical and the latter was proved independently by Bodirsky and Sokić.)

In joint work in progress of Krzysztof Krupiński, we have been generalizing the constructions above, as well as generalizing Ramsey-theoretic results involving these constructions. Our work is motivated by the recent development of the model-theoretic generalization of Ramsey theory and topological dynamics by Hrushovski, Krupiński, Lee, Moconja, and Pillay, and attempts to answer a few open questions in the area.

Surprisingly, this work proved itself to have significant results in classical structural Ramsey theory, properly and uniformly generalizing many of the most recent results in the field.

In this talk, I will present the motivation, open problems in the area, and our results.

A note on NSOP classes

Ioannis Eleftheriadis
University of Cambridge

In recent years Shelah's classification program has found surprising applications in the context of classes of relational structures. The work of [1], [2], [3] and others has explored the relationship between model-theoretic tameness, in particular (monadic) stability and (monadic) NIP, and algorithmic tameness, e.g. nowhere density and bounded twin-width. Motivated by this, we investigate the implications of NSOP in classes of graphs and relational structures. These are precisely the classes such that all ultraproducts over them have no interpretable partial order with an infinite chain.
This is a joint work with Samuel Braunfeld and Aristomenis-Dionysios Papadopoulos.

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Saturday 11th

On Rado conditions and partition regularity nonlinear Diophantine equations

Martino Lupini
Victoria University of Wellington

I will present some necessary conditions for partition regularity of Diophantine equations that generalize the classical Rado condition from the linear case. Such conditions are also sufficient in the case of certain families of degree 2 polynomials. These results have been obtained jointly with Barrett and Moreira by applying nonstandard methods and recurrence in topological dynamics, building on previous work of Di Nasso and Luperi-Baglini.

The uniform boundedness principle for (ultra)filters

Ben De Bondt
Université Paris Cité

Let X be an infinite dimensional Banach space. We formulate for each filter $\mathcal{F}$ on the set of natural numbers a corresponding uniform boundedness principle (UBP) in X, which can either hold true or fail, depending on the combinatorial properties of the filter $\mathcal{F}$. For the Fréchet filter of cofinite sets, this UBP follows from the classic Banach-Steinhaus theorem. More surprisingly, it follows from a theorem by Benedikt [1] formulated in the language of nonstandard analysis, that this UBP holds as well for selective ultrafilters. We will discuss joint work with Hans Vernaeve published in [2] which gives a combinatorial characterization of those filters $\mathcal{F}$ for which this filter version of the uniform boundedness principle holds.
This talk is based on joint work with Hans Vernaeve.

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Ultrafunctions: a new kind of generalized functions

Vieri Benci

Università di Pisa

Ultrafunctions are a particular class of functions defined on a hyperfinite grid. In this talk we present this notion of ultrafunction and their basic properties. Some applications are given to show how ultrafunctions can be applied in studying Partial Differential Equations. In particular, it is possible to prove the existence of ultrafunction solutions to ill posed evolution problems.

Congruence of ultrafilters

Boris Šobot

Faculty of Sciences, University of Novi Sad

As usual, let $\beta\mathbb{N}$ denote the set of ultrafilters on the set $\mathbb{N}$ of natural numbers, with elements of $\mathbb{N}$ identified with principal ultrafilters. A quasiorder $\widetilde{\mid}$ on $\beta\mathbb{N}$, a natural extension of the divisibility relation $\mid$ on $\mathbb{N}$, was considered in several papers ([3],[4]). It turned out that nonstandard methods can lead to better understanding of this relation ([5],[6]).
One can also extend to ultrafilters the congruence relation modulo an integer, and we examine to what extent this relation agrees with $\widetilde{\mid}$. Afterwards we propose a way to define congruence modulo ultrafilter and find its nonstandard characterization. Using iterated nonstandard

extensions, and in particular the notion of tensor pairs ([1],[2]), we also introduce so-called strong congruence. This relation is perhaps less natural, but has nicer properties with respect to $\widetilde{\mid}$. Finally, this leads to a natural strengthening of the divisibility of ultrafilters.

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Extra

This chapter contains the abstracts of those invited or contributed speakers who were not able to come to Pisa.

Ultrafilters Containing Intersections

Andreas Blass

Mathematics Department, University of Michigan

I plan to discuss several properties of ultrafilters $\mathcal{U}$ of the form: Any sufficiently large family of sets has a moderately large subfamily $\mathcal{X}$ such that a single $A \in \mathcal{U}$ decides every $X \in \mathcal{X}$. Here "A decides X" means that either $A \subseteq X$ (and so $X \in \mathcal{U}$) or $A \cap X = \varnothing$ (and so $X \notin \mathcal{U}$). In other words, if we replace some of the sets in $\mathcal{X}$ by their complements, so that all are in $\mathcal{U}$, then the intersection of them all is also in $\mathcal{U}$.
Different choices of "sufficiently large" and "moderately large" produce some interesting properties and some open problems

- An ultrafilter $\mathcal{U}$ is **not** at the top of the Tukey ordering (of continuum-sized directed sets) iff every family of $2^{\aleph_0}$ sets has an infinite subfamily decided by a single set.
- An ultrafilter $\mathcal{U}$ is preserved by some forcing that adds reals iff every perfect family has an uncountable subfamily decided by a single set.

I'll give the relevant background information about the Tukey order and about preservation by forcing.
All P-points have both of these properties, and so do all ultrafilters obtained from P-points by (iterated) summation. But there are models of ZFC without P-points, and it is open whether ZFC proves the existence of ultrafilters with either or both of these properties.
Furthermore, ZFC does not prove that P-points satisfy the common generalization that every uncountable family has an uncountable subfamily decided by a single set.

A nonstandard proof of the spectral theorem for unbounded self-adjoint operators

Isaac Goldbring
University of California

The spectral theorem for Hermitian matrices states that, given an $n \times n$ Hermitian matrix A, if one lists its distinct eigenvalues as $\lambda_1, \ldots, \lambda_k$ and lets $P_1, \ldots, P_k$ denote the orthogonal projections onto the corresponding eigenspaces, then $P_1 + \cdots + P_k$ represents an orthogonal decomposition of the identity operator on $\mathbb{C}^n$ and one has the spectral resolution $A = \sum_{i=1}^{k} \lambda_i P_i$ of A. An important generalization of this result is to the case when one considers infinite-dimensional Hilbert spaces and unbounded self-adjoint operators on this space. In this setting, one replaces finite orthogonal decompositions of the identity by projection-valued measures supported on the spectrum of the operator and the above resolutions of the identity and the operator itself appear in terms of appropriate integrals with respect to these projection-valued measures. The unbounded version of the spectral theorem is of central importance in quantum mechanics, where it is used to provide probability distributions for measurements of observables with continuous spectrum such as position and momentum.

The usual proof of the spectral theorem for unbounded self-adjoint operators usually proceeds by a reduction, via the Cayley transform, to the case of bounded, normal operators on infinite-dimensional Hilbert spaces. Thus, besides needing to first generalize to the intermediate case of bounded operators on infinite-dimensional Hilbert spaces, this approach suffers from the need to leave the realm of self-adjoint operators and work with the larger class of normal operators as well as from the need to motivate the use of the Cayley transform and all of the preliminaries associated with this operation.

In this talk, we present a proof of the spectral theorem for unbounded self-adjoint operators that solely relies on the finite-dimensional version discussed above. The proof uses nonstandard analysis and uses a hyperfinite-dimensional Hilbert space that contains the standard infinite-dimensional space in an appropriate way. An approach of this kind was used by Moore to give a nonstandard proof of the spectral theorem for bounded self-adjoint operators. Added difficulty arises in the unbounded context in the form of defining the nonstandard hull of an internal operator on a hyperfinite-dimensional Hilbert space whose internal operator norm is not necessarily a finite hyperreal number. We borrow from and add upon the ideas of Raab in his work on a nonstandard approach to quantum mechanics to deal with this issue.

An application of linear formulas to the fundamental theorem of ultraproducts

Thodsaporn Kumduang
Department of Mathematics, Faculty of Science, Chiang Mai University

Linear formulas are formal expressions that are determined by linear terms, terms in which no variable occurs more than once, relation symbols of the arity n_j, logical connectors $\neg$ and $\vee$, and a quantifier $\exists$. In this work, applying linear formulas, the fundamental theorem of ultraproducts is stated. We then give a proof of such theorem by induction on the complexity of linear formulas. Some open problems and possible directions for further research are given.

Products of sequentially compact spaces and compactness with respect to a set of filters

Paolo Lipparini
Università di Tor Vergata, Roma

The notion of ultrafilter convergence has played a key role in the study of products of topological spaces; see, e. g., the survey [S]. In [K] Kombarov introduced a local notion of ultrafilter convergence, where, by "local", we mean that the ultrafilter depends on the sequence intended to converge, rather than being fixed in advance. Kombarov put in a general setting former ideas, for example, the theorem that a topological space X is countably compact if and only if every countable sequence D-converges, for some uniform ultrafilter D over ω (notice that if we assume that there is an ultrafilter D which works for every sequence, we get a stronger notion, to the effect that every power of X is countably compact!).
In [L] we extended Kombarov notion to filters, and showed that this is a proper generalization, since, for example, sequential compactness can

be characterized in terms of such a "local" filter convergence, but all the filters involved, in this case, are necessarily not maximal [L, Section 5] (notice that the notion of D-compactness formally makes sense even when D is not maximal; however, in this sense, no T_2 space with at least two points is D-compact).
The above considerations lead to results showing that, under suitably general formulations, covering properties, accumulation properties and filter convergence are all sides of the same coin. Characterizations of topological properties of a product by means of subproducts follow. For example, a product of topological spaces is sequentially compact if and only if all subproducts by $\leq \mathfrak{s}$ factors are sequentially compact (where $\leq \mathfrak{s}$ is the *splitting number*). See arXiv:1303.0815 for more.

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Structural Limits and Practical Ultrafilter

Jaroslav Nešetřil
Charles University Praha

Structural limits (called modelings) of a sequence of finite graphs are totally Borel graphs which encode the limits of satisfaction probabilities of all FO formulas of graphs in the sequence. If all these probabilities (for formulas from a fragment X) have limit then we speak about a converging sequence of finite graphs.
Those monotone classes of graphs where every converging sequence of graphs has a modeling can be characterized in an interesting way.
We outline the main steps of the proof.
This is a joint work with Patrice Ossona De Mendez.

Asymptotic fixed points and theorems of Du Bois-Reymond and Hadamard

David Ross
University of Hawaii

Some existence theorems for asymptotic fixed points have a variety of short nonstandard proofs, for example a natural one using the nonstandard hull and a slightly trickier one that avoids the hull construction but instead uses facts about the order structure of the infinite integers.

This talk is motivated by the observation that 19th century theorems of Paul Du Bois-Reymond and Jacques Hadamard about scales of functions resemble permanence and $\aleph_1$ saturation principals from nonstandard analysis. I'll show how that resemblance can be used to translate a nonstandard fixed-point proof into a standard one.

Between the Rudin–Keisler and Comfort preorders

Denis I. Saveliev
Institute for Information Transmission Problems RAS

We consider ultrafilters over ω (although most of our results remain true for ultrafilters over any infinite set); as usual, $\beta\omega$ denotes the set of them. For $\mathfrak{u}, \mathfrak{v} \in \beta\omega$ and any ordinal α, define: $\mathfrak{u}\, R_0\, \mathfrak{v}$ iff $\mathfrak{u}$ is principal, $R_{<\alpha} = \bigcup_{\beta<\alpha} R_\beta$, and $\mathfrak{u}\, R_\alpha\, \mathfrak{v}$ iff there exists a continuous map $f : \beta\omega \to \beta\omega$ such that $f(\mathfrak{v}) = \mathfrak{u}$ and $f(n)\, R_{<\alpha}\, \mathfrak{v}$ for all $n < \omega$. The hierarchy is non-degenerate and lies between $\leq_{\mathrm{RK}}$ and $\leq_{\mathrm{C}}$, the Rudin–Keisler and Comfort preorders.

Theorem 1. *$R_1 = {\leq_{\mathrm{RK}}}$; $R_{<\alpha} \subset R_\alpha$ for all $\alpha < \omega_1$; $R_{<\omega_1} = R_{\omega_1} = {\leq_{\mathrm{C}}}$.*

If $n < \omega$, the relations R_n can be redefined in terms of right-continuous ultrafilter extensions of n-ary operations on ω as follows: $\mathfrak{u}\, R_n\, \mathfrak{v}$ iff there exists $f : \omega^n \to \omega$ such that $\widetilde{f}(\mathfrak{v}, \ldots, \mathfrak{v}) = \mathfrak{u}$. Moreover, $R_m \circ R_n = R_{nm}$ (so R_n are not preorders for $2 \leq n < \omega$). These observations can be expanded to all R_α by using ω-ary operations. Such an operation is identified with a continuous map of the Baire space ω^ω into the discrete space ω; these maps admit a natural hierarchy ranked by countable ordinals. Any continuous $f : \omega^\omega \to \omega$ uniquely extends to a right-continuous $\widetilde{f} : (\beta\omega)^\omega \to \beta\omega$, i.e., an ω-ary operation on $\beta\omega$.

Proposition 1. *Let $\alpha < \omega_1$ and $\mathfrak{u}, \mathfrak{v} \in \beta\omega$. Then $\mathfrak{u}\, R_\alpha\, \mathfrak{v}$ iff there exists a continuous $f : \omega^\omega \to \omega$ of rank α such that $\widetilde{f}(\mathfrak{v}, \mathfrak{v}, \ldots) = \mathfrak{u}$.*

The composition of arbitrary $R_{<\alpha}$ is expressed via a multiplication-like operation on ordinals. To simplify notation, denote $\sup_{\gamma<\alpha}(\gamma \cdot \beta)$ by $({<}\alpha) \cdot \beta$; the explicit calculation of these ordinals, used in getting the following result, is rather cumbersome.

Theorem 2. *Let $\alpha, \beta < \omega_1$.*

(i) *$R_\alpha \circ R_\beta = R_\gamma$ where $\gamma = \beta \cdot \alpha$ if $\beta = 0$ or $\alpha < \omega$, $\gamma = \beta \cdot (\alpha+1) - 1$ if $0 < \beta < \omega$ and $\alpha \geq \omega$, and $\gamma = \beta \cdot (\alpha + 1)$ if $\alpha, \beta \geq \omega$;*

(ii) *If $\alpha > 0$ is limit, then $R_{<\alpha} \circ R_\beta = R_{<\gamma}$ where $\gamma = \beta \cdot \alpha$;*

(iii) *If $\beta > 0$ is limit, then $R_\alpha \circ R_{<\beta} = R_{<\gamma}$ where $\gamma = ({<}\beta) \cdot \alpha$ if $\alpha < \omega$, and $\gamma = ({<}\beta) \cdot (\alpha + 1)$ otherwise;*

(iv) If $\alpha, \beta > 0$ are limit, then $R_{<\alpha} \circ R_{<\beta} = R_{<\gamma}$ where $\gamma = ({<}\beta) \cdot \alpha$.

Corollary 1. *Let $2 \leq \alpha \leq \omega_1$. Then $R_{<\alpha}$ is a preorder iff α is multiplicatively indecomposable.*

Define preorders between $\leq_{\mathrm{RK}}$ and $\leq_{\mathrm{C}}$ by letting $\leq_0 = \leq_{\mathrm{RK}}$ and $\leq_{1+\alpha} = R_{<\omega^{\omega^\alpha}}$ for all $\alpha \leq \omega_1$. So, if α is infinite, $R_{<\alpha} = \leq_\alpha$ iff α is an epsilon number. Also $\leq_\alpha \circ \leq_\beta = \leq_\gamma$ where $\gamma = \max(\alpha, \beta)$.
As was known, for any ultrafilter $\mathfrak{v}$ and semigroup S, the set $\{\mathfrak{u} : \mathfrak{u} \leq_{\mathrm{C}} \mathfrak{v}\}$ forms a subsemigroup of βS. This can be expanded to arbitrary first-order models and relations $R_{<\alpha}$ as follows.

Corollary 2. *For every $\alpha > 1$, ultrafilter $\mathfrak{v}$, and model $\mathfrak{A}$ of any signature, $\{\mathfrak{u} : \mathfrak{u}\, R_{<\alpha}\, \mathfrak{v}\}$ forms a submodel of the model $\beta\mathfrak{A}$ iff α is additively indecomposable. Consequently, for all $\alpha > 0$, $\mathfrak{v}$, and $\mathfrak{A}$, $\{\mathfrak{u} : \mathfrak{u} \leq_\alpha \mathfrak{v}\}$ forms a submodel of $\beta\mathfrak{A}$.*

Ultrafilter extensions of ω-ary operations can be used to state Ramsey-type results. Let $f[X]$ be the image of X under f, and let $I = \{x \in \omega^\omega : x \text{ is increasing}\}$. If $X \subseteq \omega$ and $f : \omega^\omega \to Y$, we say that f is *constant upward on* X iff $|f[X^\omega \cap I]| = 1$, and *quasi-invertible upward on* X iff there exists $g : Y \to \omega$ such that for any infinite $A \subseteq X$ we have $g[f[A^\omega \cap I]] \subseteq A$ and $|A \setminus g[f[A^\omega \cap I]]| < \omega$. The following refines the well-known characterization of Ramsey ultrafilters as selective ones.

Proposition 2. *A non-principal $\mathfrak{u} \in \beta\omega$ is $\leq_{\mathrm{RK}}$-minimal iff any continuous $f : \omega^\omega \to \omega$ is either constant upward or quasi-invertible upward on some $X \in \mathfrak{u}$.*

This is a joint work with Nikolai L. Poliakov.

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